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1989 J. Phys. A: Math. Gen. 22 1883

(http://iopscience.iop.org/0305-4470/22/11/026)

J. Phys. A: Math. Gen. 22 (1989) 1883-1897. Printed in the UK

Analytical evaluation of a class of lattice sums in arbitrary dimensions

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Received 24 May 1988, in final form 14 September 1988

Abstract. Starting from the Poisson summation formula in m dimensions, a class of lattice sums is evaluated analytically. The resulting formulae provide a considerable generalisation of the results reported previously and are applicable to a variety of physical problems, especially to the analysis of finite-size effects in systems undergoing phase transitions.

1. Introduction

In recent studies of finite-size effects in systems undergoing phase transitions (Singh and Pathria 1985a, b, 1986a, b, 1987a, b, c), we have encountered a variety of lattice sums which fall under the category

$$\mathscr{K}(\nu|m; y) = \sum_{\boldsymbol{q}(m)}{}' \frac{K_\nu(2yq)}{(yq)^\nu} \qquad y>0,\ q=(q_1^2+\cdots+q_m^2)^{1/2}>0 \tag{1}$$

where $K_\nu(z)$ are modified Bessel functions, y is a *scaled* parameter (which represents some characteristic physical dimension of the system in terms of its correlation length), while the summation goes over all integral components of the m-dimensional vector $\boldsymbol{q}$—excluding the term with $\boldsymbol{q}=0$. A knowledge of the analytical behaviour of these sums in different domains of the parameter y is vital in understanding the physical behaviour of the given system in different domains of the temperature variable T; this is especially true of the case $y \ll 1$ which, for most systems, corresponds to the regime $T<T_c$, T_c being the critical temperature of the corresponding bulk system. In the close neighbourhood of T_c, as T varies from values $\lesssim T_c$ to values $\gtrsim T_c$, the parameter y varies rapidly from being much less than 1 to becoming much greater than 1; at the same time, the nature of the finite-size effects in the system also changes radically as we move from the region of the first-order phase transition ($T<T_c$) to the region of the second-order phase transition ($T \simeq T_c$). A complete study of this phenomenon requires detailed information on the properties of the sums (1) over the entire range of y. This information can be acquired with the help of the Poisson summation formula—a technique employed earlier to study sums which turn out to be special cases ($\nu=0, \pm\frac{1}{2}$) of the ones presently under consideration (see, for instance, Chaba and Pathria 1975, 1976a, b, 1977, Zasada and Pathria 1976).

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0305-4470/89/111883+15$02.50

Related sums have appeared in several other problems such as the electronic-structure studies of crystalline solids (Sholl 1966, 1967, Harris and Monkhorst 1970), the analysis of the stability of quantised vortex arrays in type-II superconductors and in rotating superfluid helium (Fetter 1966, 1975, Fetter *et al* 1966), the study of the critical behaviour of ferromagnetic films (Barber and Fisher 1973), etc. Analytical attempts at unravelling the properties of these sums have been made by several authors—notably by Fisher and Barber (1972) and by Glasser and Zucker (1980). While the emphasis of the former has been on a detailed analysis of one-dimensional sums, the latter have concentrated on special sums that can be evaluated exactly. Our approach, on the other hand, has been to establish identities which relate the original, often slowly convergent, sums to new, fast converging, ones which in turn enable us to carry out an incisive study of the physical situation at hand. In that very spirit we undertook a detailed analysis of the sums defined in (1), the results of which are being reported here. While some of these results (such as the asymptotic limits for $y \to 0$ or $y \to \infty$) have appeared on occasion, most of them are new and provide a considerable generalisation of the ones reported earlier.

Our starting point in this analysis is the Poisson identity

$$\sum_{l=-\infty}^{\infty} \exp(-\lambda l^2) = \left(\frac{\pi}{\lambda}\right)^{1/2} \sum_{q=-\infty}^{\infty} \exp(-\pi^2 q^2/\lambda) \qquad 0<\lambda<\infty \tag{2}$$

whose m-dimensional version may be written as

$$1+\sum_{l(m)}{}' \exp(-\lambda l^2) = \left(\frac{\pi}{\lambda}\right)^{m/2}\left(1+\sum_{q(m)}{}' \exp(-\pi^2 q^2/\lambda)\right) \qquad m=1,2,3,\ldots \tag{3}$$

where the primed summation over l or q implies that the term pertaining to the origin of the lattice is excluded. Multiplying (3) by $\lambda^{\frac{1}{2}m-1-\nu}$ and integrating over λ, we obtain another identity, viz

$$\frac{\lambda^{\frac{1}{2}m-\nu}}{(\frac{1}{2}m-\nu)} - \sum_{l(m)}{}' \frac{\Gamma(\frac{1}{2}m-\nu, \lambda l^2)}{l^{m-2\nu}} + C(\nu|m) = -\frac{\pi^{m/2}}{\nu\lambda^{\nu}} + \pi^{\frac{1}{2}m-2\nu} \sum_{q(m)}{}' \frac{\Gamma(\nu, \pi^2 q^2/\lambda)}{q^{2\nu}} \qquad \nu \neq 0, \tfrac{1}{2}m \tag{4}$$

where $\Gamma(a, z)$ denotes the incomplete gamma function while $C(\nu|m)$ is a constant of integration which will be determined later. From here on, the analysis depends crucially on the value of the index ν.

2. Lattice sums with $\nu < 1$

Taking the Laplace transform

$$F(p) = \int_0^{\infty} e^{-p\lambda} f(\lambda)\, d\lambda \qquad p>0 \tag{5}$$

of (4), we obtain

$$\frac{\Gamma(\frac{1}{2}m-\nu)}{p^{\frac{1}{2}m-\nu+1}} - \frac{\Gamma(\frac{1}{2}m-\nu)}{p} \sum_{l(m)}{}' \frac{1}{l^{m-2\nu}}\left[1-\left(1+\frac{p}{l^2}\right)^{\nu-\frac{1}{2}m}\right] + \frac{C(\nu/m)}{p} = \frac{\pi^{m/2}\Gamma(-\nu)}{p^{1-\nu}} + 2\pi^{\frac{1}{2}m-\nu} p^{\frac{1}{2}\nu-1} \sum_{q(m)}{}' \frac{1}{q^{\nu}} K_{\nu}(2\pi\sqrt{p}q) \tag{6}$$

which holds for all $\nu < 1$—except, of course, for $\nu = 0$ or $\frac{1}{2}m$, the latter restriction being relevant only if $m = 1$. Setting $p = y^2/\pi^2$ and rearranging, we obtain our principal results, namely

$$\mathcal{K}(\nu|m;y)=\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\nu)y^{-m}+\tfrac{1}{2}\pi^{2\nu-\frac{1}{2}m}C(\nu|m)y^{-2\nu}$$
$$-\tfrac{1}{2}\Gamma(-\nu)+\tfrac{1}{2}\pi^{2\nu-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m-\nu)y^{-2\nu}\sum_{l(m)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{\nu-\frac{1}{2}m}-l^{2\nu-m}\right] \qquad (7)$$

the restrictions on ν being the same as specified above. One readily infers that in the limit $y\to 0$ the sum in question assumes the asymptotic form

$$\mathcal{K}(\nu|m;y)\approx\begin{cases}\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\nu)y^{-m} & \nu<\tfrac{1}{2}m \qquad (8a)\\ \tfrac{1}{2}\pi^{2\nu-\frac{1}{2}m}C(\nu|m)y^{-2\nu} & \nu>\tfrac{1}{2}m. \qquad (8b)\end{cases}$$

The constant $C(\nu|m)$ appearing in (8b) can be obtained from equation (4) by letting $\lambda\to\infty$. For this we note that, with λ large but finite, the second term on the left-hand side of (4) assumes the asymptotic form (see Abramowitz and Stegun 1970)

$$\sum_{l(m)}{}'\frac{(\lambda l^2)^{(m/2)-\nu-1}\exp(-\lambda l^2)}{l^{m-2\nu}}=\lambda^{(m/2)-\nu-1}\sum_{l(m)}{}'\frac{\exp(-\lambda l^2)}{l^2}.$$

The sum over l clearly converges and, since $\nu>\tfrac{1}{2}m$, the term in question vanishes as $\lambda\to\infty$. The second term on the right-hand side of (4) involves the sum

$$\sum_{q(m)}{}'\frac{1}{q^{2\nu}}\left(\Gamma(\omega)-\int_0^{\pi^2q^2/\lambda}\exp(-t)t^{\nu-1}\,dt\right)$$
$$=\sum_{q(m)}{}'\left(\frac{1}{q^{2\nu}}\Gamma(\nu)-\left(\frac{\pi^2}{\lambda}\right)^{\nu}\int_0^1\exp(-\pi^2q^2z/\lambda)z^{\nu 1}\,dz\right).$$

In view of the fact that

$$\sum_{q(m)}{}'\exp(aq^2)\approx\left(\frac{\pi}{a}\right)^{m/2} \qquad a\ll 1$$

(see equation (3)), the above expression assumes the asymptotic form

$$\Gamma(\nu)\sum_{q(m)}{}'q^{-2\nu}-\pi^{2\nu-(m/2)}\lambda^{(m/2)-\nu}(\nu-\tfrac{1}{2}m)^{-1}.$$

The limit $\lambda\to\infty$ is now straightforward to take, with the result that

$$C(\nu|m)=\pi^{\frac{1}{2}m-2\nu}\Gamma(\nu)\sum_{q(m)}{}'q^{-2\nu} \qquad \nu>\tfrac{1}{2}m \qquad (9)$$

and hence

$$\mathcal{K}(\nu|m;y)\approx\tfrac{1}{2}\Gamma(\nu)\sum_{q(m)}{}'q^{-2\nu}y^{-2\nu} \qquad \nu>\tfrac{1}{2}m. \qquad (10)$$

It seems worthwhile to point out here that the asymptotic expression (8a) could be obtained directly from (1) by replacing the summation over $\boldsymbol{q}$ by an integration (over the m-dimensional $\boldsymbol{q}$-space), which converges only if $\nu<\tfrac{1}{2}m$; at the same time, expression (10) could be obtained by replacing the function $K_\nu(z)$ by its limiting form for $z\to 0$, leaving behind a sum that converges only if $\nu>\tfrac{1}{2}m$. The virtue of equation (7), however, is that not only does it yield both forms of the asymptotic result for $y\to 0$ but it also determines corrections to it when y is not so small; in fact, it provides a complete representation of the sum $\mathcal{K}(\nu|m;y)$ in ascending powers of y which holds for all $y>0$.

The case $\nu<0$ turns out to be exceedingly simple, though still important. The constant $C(\nu|m)$ in this case can be derived from equation (4) by letting $\lambda\to 0$; proceeding as before, we obtain

$$C(\nu|m)=\Gamma(\tfrac{1}{2}m-\nu)\sum_{l(m)}{}' l^{-(m-2\nu)} \qquad \nu<0. \tag{11}$$

Substituting (11) into (7) and writing $\nu=-\mu$ ($\mu>0$), we get

$$\sum_{q(m)}{}' (yq)^{\mu}K_{\mu}(2yq)=\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m+\mu)y^{-m}-\tfrac{1}{2}\Gamma(\mu)$$
$$+\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m+\mu)y^{2\mu}\sum_{l(m)}{}'(\pi^2l^2+y^2)^{-(\frac{1}{2}m+\mu)} \qquad \mu>0 \tag{12}$$

which may, in fact, be written in terms of 'full' sums:

$$\sum_{q(m)} (yq)^{\mu}K_{\mu}(2yq)=\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m+\mu)y^{2\mu}\sum_{l(m)}(\pi^2l^2+y^2)^{-(\frac{1}{2}m+\mu)} \qquad \mu>0. \tag{13}$$

Equation (13) provides a remarkable generalisation of the well known identity

$$\sum_{q=-\infty}^{\infty}\exp\left(-\frac{2\pi}{\alpha}|q|\right)=\coth\left(\frac{\pi}{\alpha}\right)=\frac{\alpha}{\pi}\sum_{l=-\infty}^{\infty}(\alpha^2l^2+1)^{-1} \tag{14}$$

(see Morse and Feshbach 1953), which is just a special case ($\mu=\tfrac{1}{2}$, $m=1$) of our result, with $y=\pi/\alpha$.

The domain $0<\nu<\tfrac{1}{2}m$ presents special problems because the constant $C(\nu|m)$ in this domain does not assume any simple form such as we have in equation (9) or (11). In practical applications, however, this turns out to be the more commonly occurring case because quite often we are confronted with situations where $0<\nu<1$, while $m=1$, 2 or 3. The evaluation of the constant $C(\nu|m)$ in the domain $0<\nu<\tfrac{1}{2}m$ is, therefore, a matter of considerable importance. In view of this, we pursued this question with some zeal and the results of that pursuit are summarised in appendix 1.

We shall now proceed to study sums with $\nu>1$, still excluding the cases where ν is integral or half-odd-integral; those special cases will be studied at length in §§ 4 and 5, respectively.

3. Lattice sums with $\nu>1$

We shall now prefer to write $\nu=n+\eta$, where $n=0,1,2,\ldots$, while $0<\eta<1$. The case $n=0$ is covered by the principal result (7), which may now be written as

$$\mathscr{K}(\eta|m;y)=\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\eta)y^{-m}+\tfrac{1}{2}\pi^{2\eta-\frac{1}{2}m}C(\eta|m)y^{-2\eta}$$
$$-\tfrac{1}{2}\Gamma(-\eta)+\tfrac{1}{2}\pi^{2\eta-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m-\eta)y^{-2\eta}\sum_{l(m)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{\eta-\frac{1}{2}m}-l^{2\eta-m}\right]. \tag{15}$$

In view of the fact that the sums in (1) satisfy the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}y}[y^{2\nu}\mathscr{K}(\nu|m;y)]=-2y^{2\nu-1}\mathscr{K}(\nu-1|m;y) \tag{16}$$

we may write

$$\mathscr{K}(\eta+1|m;y)=-\frac{2}{y^{2\eta+2}}\int y^{2\eta+1}\mathscr{K}(\eta|m;y)\,\mathrm{d}y+\frac{\text{constant}}{y^{2\eta+2}} \tag{17}$$

with a constant of integration not yet specified. Substituting (15) into (17) and carrying out the integration over y, we obtain

$$\begin{aligned}\mathcal{K}(\eta+1|m;y) &= \tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\eta-1)y^{-m}+E(\eta+1|m)y^{-2\eta-2}\\ &\quad -\tfrac{1}{2}\pi^{2\eta-\frac{1}{2}m}C(\eta|m)y^{-2\eta}-\tfrac{1}{2}\Gamma(-\eta-1)\\ &\quad +\tfrac{1}{2}\pi^{2\eta+2-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m-\eta-1)y^{-2\eta-2}\sum_{l(m)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{\eta-\frac{1}{2}m+1}\right.\\ &\quad \left. -l^{2\eta-m+2}-(\eta-\tfrac{1}{2}m+1)l^{2\eta-m}\frac{y^2}{\pi^2}\right] \end{aligned} \tag{18}$$

where the summand on the right-hand side here is so chosen as to make the sum over $l(m)$ convergent, with the result that the term involving this sum vanishes as $y^{2(1-\eta)}$ as $y\to 0$; the constant of integration is now well defined and is denoted by the symbol $E(\eta+1|m)$. Successive applications of this operation lead to the general result

$$\begin{aligned}\mathcal{K}(\eta+n|m;y) &= \tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\eta-n)y^{-m}+\sum_{l=1}^{n}\frac{(-1)^{n-l}E(\eta+l|m)}{(n-l)!}y^{-2\eta-2l}\\ &\quad +\frac{(-1)^n\pi^{2\eta-\frac{1}{2}m}C(\eta|m)}{2(n!)}y^{-2\eta}-\tfrac{1}{2}\Gamma(-\eta-n)\\ &\quad +\tfrac{1}{2}\pi^{2\eta+2n-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m-\eta-n)y^{-2\eta-2n}\sum_{l(m)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{\eta-\frac{1}{2}m+n}\right]_{(n)} \end{aligned} \tag{19}$$

where we have introduced the compact notation (cf Fisher and Barber 1972)

$$[g(p)]_{(n)}=g(p)-\sum_{r=0}^{n}\frac{g^{(r)}(0)}{r!}p^r \qquad p=y^2/\pi^2. \tag{20}$$

It is not surprising that equation (19) contains n new constants of integration which sooner or later will have to be evaluated. What is surprising, however, is that these constants are so intimately related to the constants $C(\nu|m)$ that appear in equation (4) and have been studied at length in appendix 1. A straightforward, though somewhat tedious, calculation given in appendix 2 shows that, quite generally,

$$E(\nu|m)=\tfrac{1}{2}\pi^{2\nu-\frac{1}{2}m}C(\nu|m). \tag{21}$$

In view of this, the second and third terms on the right-hand side of (19) may be combined together to write instead

$$\tfrac{1}{2}\pi^{-m/2}\sum_{l=0}^{n}\frac{(-1)^{n-l}C(\eta+l|m)}{(n-l)!}\left(\frac{\pi^2}{y^2}\right)^{\eta+l}. \tag{22}$$

A perusal of expressions (19) and (22) shows that the asymptotic behaviour of the sum $\mathcal{K}(\eta+n|m;y)$, as $y\to 0$, is given by

$$\mathcal{K}(\eta+n|m;y)\approx\begin{cases}\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\eta-n)y^{-m} & (\eta+n)<\tfrac{1}{2}m \quad (23a)\\ \tfrac{1}{2}\pi^{2\eta+2n-\frac{1}{2}m}C(\eta+n|m)y^{-2(\eta+n)} & (\eta+n)>\tfrac{1}{2}m \quad (23b)\end{cases}$$

which generalise formulae (8a) and (8b) of § 2.

At this point we should note the limitations of the results obtained in this section. These limitations arise from the conditions restricting (some of) the steps on which the derivation of these results is based but can be seen more readily by looking at the mathematical functions appearing in equation (19). We thus find that, apart from the basic restriction $0<\eta<1$, we also require that, if $\eta=\frac{1}{2}$ and m is odd, then n must be restricted to values $0, 1, \ldots, \frac{1}{2}(m-3)$; the case where $n \geqslant \frac{1}{2}(m-1)$ will be covered in § 5. However, before we proceed to that case, we shall study the more intriguing case $\eta=0$, which presents some difficulties for all values of n but more so if m is even and $n \geqslant \frac{1}{2}m$.

4. Lattice sums with integral ν

The obvious procedure to tackle this case is to take the results of the previous section and (with caution) let $\eta \rightarrow 0$. In doing so, we make use of the facts that

$$\lim_{\eta\rightarrow 0} \Gamma(-\eta-n)=\frac{(-1)^{n+1}}{n!}[1/\eta-\psi(n+1)] \tag{24}$$

where $\psi(n+1)$ is the digamma function:

$$\psi(n+1)=\frac{\mathrm{d}}{\mathrm{d}z}\ln \Gamma(z)|_{z=n+1}=-\gamma+\sum_{k=1}^{n} k^{-1} \tag{25}$$

γ being the Euler constant, while

$$\lim_{\eta\rightarrow 0}[C(\eta|m)x^{-\eta}]=-\frac{\pi^{m/2}}{\eta}+[\bar{C}(m)+\pi^{m/2}\ln x] \tag{26}$$

where $\bar{C}(m)$ is given by equation (7) of appendix 1; see also equations (A1.15), (A1.19), (A1.21), (A1.24) and (A1.26). After some algebra, we obtain

$$\begin{aligned}\mathscr{K}(n|m;y)=&\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-n)y^{-m}+\tfrac{1}{2}\pi^{-m/2}\sum_{l=1}^{n}\frac{(-1)^{n-l}C(l|m)}{(n-l)!}\left(\frac{\pi^2}{y^2}\right)^{l}\\&+\frac{(-1)^n}{2(n!)}[\pi^{-m/2}\bar{C}(m)+\ln(y^2/\pi^2)-\psi(n+1)]\\&+\tfrac{1}{2}\pi^{2n-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m-n)y^{-2n}\sum_{l(m)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{n-\frac{1}{2}m}\right]_{(n)}\end{aligned} \tag{27}$$

which holds (i) for all n if m is odd and (ii) for $n<\frac{1}{2}m$ if m is even. The asymptotic behaviour of $\mathscr{K}(n|m;y)$, as $y\rightarrow 0$, is given by

$$\mathscr{K}(n|m;y)\approx\begin{cases}\frac{1}{2}\pi^{m/2}\Gamma(\frac{1}{2}m-n)y^{-m} & n<\frac{1}{2}m \quad (28a)\\ \frac{1}{2}\pi^{2n-m/2}C(n|m)y^{-2n} & n>\frac{1}{2}m \quad (28b)\end{cases}$$

which are essentially the same as (23).

The case with m even and $n \geqslant \frac{1}{2}m$ demands extra care because now fresh trouble arises from several other terms of the general result (19). To begin with, we replace m by $2s$, where $s=1, 2, 3, \ldots$, set $n=s$ and let $\eta\rightarrow 0$. Apart from equations (24)-(26), we now make use of the following as well:

$$\lim_{\eta\rightarrow 0}\Gamma(-\eta)=-1/\eta-\gamma \tag{29}$$

$$\lim_{\eta\rightarrow 0}[C(\eta+s|2s)x^{-\eta}]=1/\eta+[\pi^{-s}\bar{C}(2s)-2\ln \pi-\ln x] \tag{30}$$

(see equation (6) of appendix 1) and

$$\lim_{\eta\to 0}\Gamma(-\eta)[(l^2+y^2/\pi^2)^\eta - l^{2\eta}] = -\ln(1+y^2/\pi^2 l^2). \tag{31}$$

We thus obtain

$$\begin{aligned}\mathcal{K}(s|2s;y) &= \tfrac{1}{2}\pi^s y^{-2s}[-\gamma+\pi^{-s}\bar{C}(2s)-\ln(y^2)] + \tfrac{1}{2}\pi^{-s}\sum_{l=1}^{s-1}\frac{(-1)^{s-l}C(l|2s)}{(s-l)!}\left(\frac{\pi^2}{y^2}\right)^l \\ &\quad + \frac{(-1)^s}{2(s!)}[\pi^{-s}\bar{C}(2s)+\ln(y^2/\pi^2)-\psi(s+1)] \\ &\quad - \tfrac{1}{2}\pi^s y^{-2s}\sum_{l(2s)}{}' \left[\ln\left(1+\frac{y^2}{\pi^2 l^2}\right)\right]_{(s)}.\end{aligned} \tag{32}$$

The asymptotic behaviour of this sum is given by

$$\mathcal{K}(s|2s;y) \approx \tfrac{1}{2}\pi^s y^{-2s}\{\ln(1/y^2)+[\pi^{-s}\bar{C}(2s)-\gamma]\} \tag{33}$$

which is characteristically different from (28*a*) and (28*b*); in fact, it is a curious amalgam of the two.

Finally, a repeated application of the recurrence formula (16) to equation (32) yields the more general result:

$$\begin{aligned}\mathcal{K}(s+s'|2s;y) &= \frac{(-1)^{s'}\pi^s}{2(s'!)}y^{-2s}[\psi(s'+1)+\pi^{-s}\bar{C}(2s)-\ln(y^2)] \\ &\quad + \tfrac{1}{2}\pi^{-s}\sum_{\substack{l=1\\(l\neq s)}}^{s+s'}\frac{(-1)^{s+s'-l}C(l|2s)}{(s+s'-l)!}\left(\frac{\pi^2}{y^2}\right)^l \\ &\quad + \frac{(-1)^{s+s'}}{2(s+s')!}[\pi^{-s}\bar{C}(2s)+\ln(y^2/\pi^2)-\psi(s+s'+1)] \\ &\quad - \frac{(-1)^{s'}\pi^{s+2s'}}{2(s'!)}y^{-2s-2s'}\sum_{l(2s)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{s'}\ln\left(1+\frac{y^2}{\pi^2 l^2}\right)\right]_{(s+s')}\end{aligned} \tag{34}$$

where $s'=0, 1, 2, \ldots$. The asymptotic behaviour of this sum, for $s'>0$, is given by

$$\mathcal{K}(s+s'|2s;y) \approx \tfrac{1}{2}\pi^{s+2s'}C(s+s'|2s)y^{-2(s+s')} \qquad s'>0 \tag{35}$$

which is consistent with (28*b*).

5. Lattice sums with half-odd-integral ν

In the end we consider the case $\nu = n+\frac{1}{2}$, where $n=0,1,2,\ldots,$. If m is even, this case is already covered by equation (19), with $\eta=\frac{1}{2}$; the same is true if m is odd and $n<\frac{1}{2}(m-1)$. The only situation remaining to be considered here is the one pertaining to m odd and $n\geq\frac{1}{2}(m-1)$. Writing $m=2s-1$ $(s=1,2,3,\ldots)$ and $n=\frac{1}{2}(m-1)+s'$

($s'=0, 1, 2, \ldots$), and following the standard procedure, we now obtain

$$\mathscr{K}(s+s'-\tfrac{1}{2}|2s-1;y)$$
$$=\frac{(-1)^{s'}\pi^{s-\frac{1}{2}}}{2(s'!)}y^{-(2s-1)}[\psi(s'+1)+\pi^{-(s-\frac{1}{2})}\bar{C}(2s-1)-\ln(y^2)]$$
$$+\tfrac{1}{2}\pi^{-(s-\frac{1}{2})}\sum_{\substack{l=1\\(l\neq s)}}^{s+s'}\frac{(-1)^{s+s'-l}C(l-\frac{1}{2}|2s-1)}{(s+s'-l)!}\left(\frac{\pi^2}{y^2}\right)^{l-\frac{1}{2}}-\tfrac{1}{2}\Gamma(-s-s'+\tfrac{1}{2})$$
$$-\frac{(-1)^{s'}\pi^{s+2s'-\frac{1}{2}}}{2(s')!}y^{-2s-2s'+1}\sum_{l(2s-1)}{}'\left[\left(l^2+\frac{y^2}{\pi^2}\right)^{s'}\ln\left(1+\frac{y^2}{\pi^2 l^2}\right)\right]_{(s+s'-1)}. \quad (36)$$

The asymptotic behaviour of this sum, as $y\to 0$, is given by

$$\mathscr{K}(s+s'-\tfrac{1}{2}|2s-1;y)$$
$$\approx\begin{cases}\tfrac{1}{2}\pi^{s-\frac{1}{2}}y^{-(2s-1)}\{\ln(1/y^2)+[\pi^{-(s-\frac{1}{2})}\bar{C}(2s-1)-\gamma]\} & s'=0 \quad (37a)\\ \tfrac{1}{2}\pi^{s+2s'-\frac{1}{2}}C(s+s'-\tfrac{1}{2}|2s-1)y^{-(2s+2s'-1)} & s'>0 \quad (37b)\end{cases}$$

which compares favourably with the one given by equations (33) and (35) of the previous section.

6. Discussion of results and concluding remarks

In the preceding sections we derived a number of results for the sums $\mathscr{K}(\nu|m;y)$, for *general* ν and m, which enable us to study the asymptotic behaviour of these sums—with corrections to *all* orders in y^2. Certain special cases of these results have already appeared in connection with the studies mentioned in the introduction. By and large, they corresponded to $\nu=-\frac{1}{2}$, 0 or $\frac{1}{2}$ and $m=1, 2$ or 3; a very special situation arose in our earliest work on this topic (see Chaba and Pathria 1975) which pertained to $\nu=\frac{1}{2}m-1$, with $m=1$, 2, 3 or 4. More recently, we have encountered situations in which ν varies continuously over the range $(-1, 1)$ and occasionally goes as far as $\pm\frac{3}{2}$ (see Singh and Pathria 1987c); this necessitated the detailed analysis whose results have been reported in the present paper.

While the results presented here are fairly general, it is heartening to note that, wherever comparison with previous work is possible, complete agreement is found. Of especial interest in this context is the work of Fisher and Barber (1972) who, in their study of finite-size effects in ferromagnetic films ($m=1$), carried out an exhaustive analysis of the sums appearing on the right-hand side of our equations—which they termed as ‘remnant functions’. They treated these functions using very different mathematical techniques; their final results, however, are precisely the same as ours, except for an understandable rearrangement of terms. We may, nonetheless, emphasise the fact that, using our method, the analysis of such functions can be carried out as effectively for $m>1$ as for $m=1$.

At this stage we would like to make a few remarks on the sums with $\nu<0$, namely those given by equations (12) and (13) where ν has been replaced by $-\mu$, so that $\mu>0$; for asymptotic purposes, we may concentrate on equation (12) only. Now, sums of this type have been encountered previously by Fetter (1966) in connection with his analysis of the stability of quantised vortex arrays in type-II superconductors and in superfluid helium. One of his results (in our notation) is

$$\sum_{q(2)}' q^2 K_2(2yq) = \frac{\pi}{y^4} - \frac{1}{2y^2} + O(y^2). \tag{38}$$

Setting $\mu = m = 2$ in (12), we obtain

$$\sum_{q(2)}' q^2 K_2(2yq) = \frac{\pi}{y^4} - \frac{1}{2y^2} + \frac{y^2}{\pi^5} \sum_{l(2)}' \left(l^2 + \frac{y^2}{\pi^2}\right)^{-3} \tag{39}$$

where terms to *all* orders in y^2 can be evaluated in a closed form using Hardy sums (see Zucker 1974)

$$\sum_{l(2)}' l^{-2s} = 4\zeta(s)\beta(s) \qquad s > 1. \tag{40}$$

Equation (39) is clearly an improvement over (38). Another sum appearing in Fetter's work is

$$\sum_{q(2)}' q^2 K_0(2yq) = \frac{\pi}{2y^4} + O(1). \tag{41}$$

We readily observe that this sum can be obtained by direct differentiation of (12), with $\mu = 1$ and $m = 2$; thus

$$\sum_{q(2)}' q^2 K_0(2yq) = \frac{\pi}{2y^4} - \frac{1}{2\pi^3} \sum_{l(2)}' \frac{l^2 - y^2/\pi^2}{(l^2 + y^2/\pi^2)^3}. \tag{42}$$

Once again, terms to *all* orders in y^2 can be derived in a closed form.

Next we would like to show that equation (12) enables us to evaluate certain two-dimensional sums *exactly*. For this we start with the one-dimensional identity (based on equation (12), with q and l replaced by n_1), namely

$$\sum_{n_1=1}^{\infty} \left(\frac{n_1}{y}\right)^{\mu} K_\mu(2yn_1) = \tfrac{1}{4}\pi^{1/2}\Gamma(\mu + \tfrac{1}{2}) y^{-2\mu-1} - \tfrac{1}{4}\Gamma(\mu) y^{-2\mu} + \tfrac{1}{2}\pi^{-2\mu-\frac{1}{2}}\Gamma(\mu + \tfrac{1}{2}) \sum_{n_1=1}^{\infty} \left(n_1^2 + \frac{y^2}{\pi^2}\right)^{-(\mu+\frac{1}{2})} \tag{43}$$

set $y = n_2\pi$ and sum over n_2 from 1 to ∞. This leads to the remarkable result

$$\sum_{n_{1,2}=1}^{\infty} \left(\frac{n_1}{n_2}\right)^{\mu} K_\mu(2\pi n_1 n_2) = \frac{\Gamma(\mu + \frac{1}{2})\zeta(\mu + \frac{1}{2})\beta(\mu + \frac{1}{2})}{2\pi^{\mu+\frac{1}{2}}} - \frac{\Gamma(\mu)\zeta(2\mu)}{4\pi^{\mu}} - \frac{\Gamma(\mu + \frac{1}{2})\zeta(2\mu + 1)}{4\pi^{\mu+\frac{1}{2}}} \tag{44}$$

which may also be written in terms of the constants $C(\nu|m)$ of appendix 1:

$$\sum_{n_{1,2}=1}^{\infty} \left(\frac{n_1}{n_2}\right)^{\mu} K_\mu(2\pi n_1 n_2) = \tfrac{1}{8}\pi^{\mu-\frac{1}{2}}[(C(\mu + \tfrac{1}{2}|2) - C(\mu|1) - \pi^{1/2} C(\mu + \tfrac{1}{2}|1)]. \tag{45}$$

The limiting cases $\mu \to 0$ and $\mu \to \frac{1}{2}$ yield the results

$$\begin{aligned}\sum_{n_{1,2}=1}^{\infty} K_0(2\pi n_1 n_2) &= \frac{1}{8\pi^{1/2}} [C(\tfrac{1}{2}|2) - 2\bar{C}(1) + 2\pi^{1/2} \ln \pi] \\ &= \tfrac{1}{2}\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}) - \tfrac{1}{4}\gamma + \tfrac{1}{4}\ln(4\pi)\end{aligned} \tag{46}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n}-1)} = \tfrac{1}{4}[\pi^{-1}\bar{C}(2) - \pi^{-1/2}\bar{C}(1) - \pi^{1/2}C(1|1)]$$

$$= \tfrac{1}{4}\ln[16\pi^3/\{\Gamma(\tfrac{1}{4})\}^4] - \tfrac{1}{12}\pi \tag{47}$$

which have been seen previously (Chaba and Pathria 1976b, 1977); the result for general μ, however, does not seem to have appeared before.

Finally we quote a result which brings out explicitly the manner in which a well known lattice sum diverges. For this we employ equation (11), with $\nu = -\tfrac{1}{2}\varepsilon$ $(\varepsilon > 0)$:

$$\sum_{l(m)}{}' \, l^{-(m+\varepsilon)} = C(-\tfrac{1}{2}\varepsilon|m)/\Gamma(\tfrac{1}{2}m + \tfrac{1}{2}\varepsilon). \tag{48}$$

In the limit $\varepsilon \to 0$, we obtain the desired result:

$$\lim_{\varepsilon\to 0}\left[\sum_{l(m)}{}' \, l^{-(m+\varepsilon)}\right] = \frac{1}{\Gamma(\tfrac{1}{2}m)}\left(\frac{2\pi^{m/2}}{\varepsilon} + \{\bar{C}(m) - \pi^{m/2}\psi(\tfrac{1}{2}m)\}\right). \tag{49}$$

While the case $m = 1$ yields the familiar result:

$$\lim_{\varepsilon\to 0}\zeta(1+\varepsilon) = (1/\varepsilon) + \gamma \tag{50}$$

cases other than $m = 1$ are not so familiar. Of course, the case $m = 2$ has been studied extensively by previous authors (see Glasser and Zucker 1980); the case of general m, to our knowledge, has not been tackled before.

Acknowledgments

We are grateful to the Natural Sciences and Engineering Research Council of Canada for financial support. One of us (RKP) is also grateful to the members of the Department of Physics and of Mathematical Sciences at the San Diego State University for the hospitality extended to him during his recent visit.

Appendix 1

In this appendix we analyse the constants $C(\nu|m)$ appearing in equations (4) and (7) for *general* ν and m. For this we write equation (4) in the form

$$C(\nu|m) = -\frac{\pi^{m/2}}{\nu\lambda^{\nu}} + \pi^{\frac{1}{2}m-2\nu}\sum_{q(m)}{}'\int_{\pi^2 q^2/\lambda}^{\infty}\frac{e^{-x}x^{\nu-1}}{q^{2\nu}}\,dx$$

$$-\frac{\lambda^{\frac{1}{2}m-\nu}}{(\tfrac{1}{2}m-\nu)} + \sum_{l(m)}{}'\int_{\lambda l^2}^{\infty}\frac{e^{-x}x^{\frac{1}{2}m-\nu-1}}{l^{m-2\nu}}\,dx \qquad \nu \neq 0, \tfrac{1}{2}m \tag{A1.1}$$

and observe that, like its left-hand side, the right-hand side of this equation must be independent of the parameter λ. For simplicity, we set $\lambda = \pi$ and redefine the variables in the two integrals to obtain

$$C(\nu|m) = \pi^{\frac{1}{2}m-\nu}\left(-\frac{\tfrac{1}{2}m}{\nu(\tfrac{1}{2}m-\nu)} + \int_1^{\infty} f_m(x)(x^{\nu} + x^{\frac{1}{2}m-\nu})x^{-1}\,dx\right) \qquad \nu \neq 0, \tfrac{1}{2}m \tag{A1.2}$$

where

$$f_m(x)=\sum_{l(m)}{}' \exp(-\pi x l^2). \tag{A1.3}$$

We readily obtain a *reflection formula* for $C(\nu|m)$:

$$\pi^\nu C(\nu|m)=\pi^{\frac{1}{2}m-\nu}C(\tfrac{1}{2}m-\nu|m) \tag{A1.4}$$

which relates the values of $C(\nu|m)$ for $\nu>m/4$ with those for $\nu<m/4$. It is straightforward to see that expression (9) for $\nu>m/2$ and expression (11) for $\nu<0$ indeed satisfy this formula.

Equation (A1.2) brings out very clearly the singularity in $C(\nu|m)$ as $\nu\to 0$ or $m/2$. We obtain two complementary results:

$$\lim_{\varepsilon\to 0} C(\varepsilon|m)=-\frac{\pi^{m/2}}{\varepsilon}+\bar{C}(m) \tag{A1.5}$$

and

$$\lim_{\varepsilon\to 0} C(\tfrac{1}{2}m-\varepsilon|m)=-1/\varepsilon+[\pi^{-m/2}\bar{C}(m)-2\ln\pi] \tag{A1.6}$$

where

$$\bar{C}(m)=\pi^{m/2}\left(\ln\pi-\frac{2}{m}+\int_1^\infty f_m(x)(1+x^{m/2})x^{-1}\,dx\right). \tag{A1.7}$$

Obviously, $\bar{C}(m)$ satisfies the inequality

$$\bar{C}(m)>\pi^{m/2}(\ln\pi-2/m) \tag{A1.8}$$

of which the special case $m=2$ has been noticed earlier (see Chaba and Pathria 1975).

Next we observe that since, by equation (3),

$$f_m(x)\equiv\sum_{l(m)}{}' \exp(-\pi x l^2)=\frac{1}{x^{m/2}}-1+\frac{1}{x^{m/2}}\sum_{q(m)}{}' \exp(-\pi q^2/x) \qquad x>0 \tag{A1.9}$$

and hence $0<f_m(x)<1/x^{m/2}$ for all x, the constant $C(\nu|m)$ itself satisfies the inequality

$$-\frac{\frac{1}{2}m}{\nu(\frac{1}{2}m-\nu)}\pi^{\frac{1}{2}m-\nu}<C(\nu|m)<0 \qquad 0<\nu<\tfrac{1}{2}m. \tag{A1.10}$$

For $\nu=\frac{1}{2}m-1$, the first part of this inequality has also been noticed earlier (Chaba and Pathria 1975). The second part, however, turns out to be even more important because in many physical applications the constant $C(\nu|m)$ appears in the very *first-order* corrections arising from the finiteness of the system (see, for instance, Singh and Pathria 1987a, 1988), and the fact that $C(\nu|m)$ in these cases is necessarily negative helps settle the question regarding the 'sign' of these corrections.

We shall now derive explicit expressions for $C(\nu|m)$ for certain special values of m. In the simplest case ($m=1$), equation (A1.2) reduces to

$$C(\nu|1)=\pi^{\frac{1}{2}-\nu}\left(-\frac{\frac{1}{2}}{\nu(\frac{1}{2}-\nu)}+\int_1^\infty f_1(x)(x^\nu+x^{\frac{1}{2}-\nu})x^{-1}\,dx\right) \qquad \nu\neq 0,\tfrac{1}{2}. \tag{A1.11}$$

Using Riemann's representation of the (Riemann) zeta function (see Whittaker and Watson 1927), we obtain

$$C(\nu|1)=2\pi^{\frac{1}{2}-2\nu}\Gamma(\nu)\zeta(2\nu) \qquad \nu\neq 0,\tfrac{1}{2}. \tag{A1.12}$$

For $\nu > \frac{1}{2}$, this is identical with equation (9); at the same time, for $\nu < 0$, it agrees with equation (11) (for the zeta function itself satisfies the reflection formula

$$F(1-s) = F(s) \tag{A1.13}$$

where $F(s) = \pi^{-s/2}\Gamma(\frac{1}{2}s)\,\zeta(s)$). Accordingly, we may as well write

$$C(\nu|1) = 2\Gamma(\tfrac{1}{2}-\nu)\zeta(1-2\nu) \qquad \nu \neq 0, \tfrac{1}{2}. \tag{A1.14}$$

The limits $\nu \to 0$ and $\nu \to \frac{1}{2}$ yield results consistent with equations (A1.5) and (A1.6), with

$$\bar{C}(1) = \pi^{1/2}(\gamma - 2\ln 2) \tag{A1.15}$$

where γ is the well known Euler's constant. A comparison of (A1.15) with (A1.7) yields the bonus result

$$\int_1^\infty f_1(x)(1+x^{1/2})x^{-1}\,\mathrm{d}x = 2 + \gamma - \ln(4\pi). \tag{A1.16}$$

Proceeding along similar lines and using certain results of Zucker (1974), we obtain the following.

(a) For $m = 2$

$$C(\nu|2) = 4\pi^{1-2\nu}\Gamma(\nu)\zeta(\nu)\beta(\nu)$$
$$= 4\Gamma(1-\nu)\zeta(1-\nu)\beta(1-\nu) \qquad \nu \neq 0, 1 \tag{A1.17}$$

where $\beta(\nu)$ is the analytic continuation of the Dirichlet series:

$$\beta(\nu) = \sum_{l=0}^{\infty} (-1)^l (2l+1)^{-\nu} \qquad \nu > 0. \tag{A1.18}$$

For analytical properties of the function $\beta(\nu)$, see Glasser (1973). The associated constant $\bar{C}(2)$ turns out to be

$$\bar{C}(2) = \pi[\gamma - \ln(\{\Gamma(\tfrac{1}{4})\}^4/4\pi^3)]. \tag{A1.19}$$

One constant that appears in the study of cylindrical geometries is $C(\frac{1}{2}|2)$ which is now seen to be equal to $4\pi^{1/2}\zeta(\frac{1}{2})\beta(\frac{1}{2}) \simeq -6.913\,040$.

(b) For $m = 4$

$$C(\nu|4) = 8(1-4^{1-\nu})\pi^{2(1-\nu)}\Gamma(\nu)\zeta(\nu-1)\zeta(\nu) \qquad \nu \neq 0, 2 \tag{A1.20}$$

while

$$\bar{C}(4) = \pi^2\left(1 + \frac{1}{3}\ln 4 + \frac{6}{\pi^2}\zeta'(2)\right). \tag{A1.21}$$

It may be pointed out here that, contrary to its deceitful appearance, expression (A1.20) is perfectly regular at $\nu = 1$, with $C(1|4) = -8\ln 2$; equation (A1.2) then yields the delightful result

$$\sum_{l(4)}{}' \frac{\mathrm{e}^{-\pi l^2}}{l^2} = \pi - 4\ln 2. \tag{A1.22}$$

(c) For $m = 6$

$$C(\nu|6) = \pi^{3-2\nu}\Gamma(\nu)[16\zeta(\nu-2)\beta(\nu) - 4\beta(\nu-2)\zeta(\nu)] \qquad \nu \neq 0, 3 \tag{A1.23}$$

while

$$\bar{C}(6)=\pi^3\left(\frac{3}{2}-\frac{2}{\pi^2}\zeta(3)+\frac{32}{\pi^3}\beta'(3)\right). \tag{A1.24}$$

(*d*) For $m=8$

$$C(\nu|8)=16\pi^{2(2-\nu)}[1-2^{1-\nu}+2^{2(2-\nu)}]\Gamma(\nu)\zeta(\nu-3)\zeta(\nu) \qquad \nu\neq 0,4 \tag{A1.25}$$

while

$$\bar{C}(8)=\pi^4\left(\frac{11}{6}+\frac{90}{\pi^4}\zeta'(4)\right). \tag{A1.26}$$

It is unfortunate that a similar degree of progress is seemingly impossible for $m=3$, 5, 7, ...; in those cases one has no choice but to resort to a numerical evaluation of the desired constants. The one that turns up regularly in our studies is $C(\frac{1}{2}|3)$ which we find to be about −8.913 633.

Appendix 2

In this appendix we establish a relationship between the constants $E(\eta+l|m)$ appearing in equation (19) and the constants $C(\nu|m)$ appearing in (4). For this we replace ν in (4) by $\eta+k$, where $0<\eta<1$ and $k=1,2,3,\ldots$, multiply by λ^k and repeat the steps that led to equation (7). In the present case we obtain a more elaborate result, namely

$$\begin{aligned}\sum_{s=0}^{k}\frac{1}{s!}\mathcal{K}(\eta+k-s|m;y)\\ &=\tfrac{1}{2}\pi^{m/2}\{\Gamma(\tfrac{1}{2}m-\eta+1)/[k!(\tfrac{1}{2}m-\eta-k)]\}y^{-m}\\ &\quad+\tfrac{1}{2}\pi^{2\eta+2k-\frac{1}{2}m}C(\eta+k|m)y^{-2\eta-2k}+\tfrac{1}{2}\Gamma(1-\eta)/[k!(\eta+k)]\\ &\quad+\tfrac{1}{2}\pi^{-m/2}p^{-\eta-k}\sum_{l(m)}{}'\left(\sum_{s=0}^{k}\frac{1}{s!}\Gamma(\tfrac{1}{2}m-\eta-k+s)p^s(l^2+p)^{\eta+k-\frac{1}{2}m-s}\right.\\ &\quad\left.-\Gamma(\tfrac{1}{2}m-\eta-k)l^{2\eta+2k-m}\right)\end{aligned} \tag{A2.1}$$

where $p=y^2/\pi^2$. Now, the function appearing on the left-hand side of this equation can also be constructed directly on the basis of equation (19), with $n=k-s$; this gives instead

$$\begin{aligned}\sum_{s=0}^{k}\frac{1}{s!}\left(\tfrac{1}{2}\pi^{m/2}\Gamma(\tfrac{1}{2}m-\eta-k+s)y^{-m}+\sum_{l=1}^{k-s}\frac{(-1)^{k-s-l}E(\eta+l|m)}{(k-s-l)!}y^{-2\eta-2l}\right.\\ +\frac{1}{2}\frac{(-1)^{k-s}\pi^{2\eta-\frac{1}{2}m}}{(k-s)!}C(\eta|m)y^{-2\eta}-\tfrac{1}{2}\Gamma(-\eta-k+s)\\ \left.+\tfrac{1}{2}\pi^{-m/2}p^{-\eta-k+s}\Gamma(\tfrac{1}{2}m-\eta-k+s)\sum_{l(m)}{}'[(l^2+p)^{\eta-\frac{1}{2}m+k-s}]_{(k-s)}\right).\end{aligned} \tag{A2.2}$$

We shall now compare expressions (A2.1) and (A2.2), which should indeed be equal.

In view of the fact that

$$\sum_{s=0}^{k}\frac{1}{s!}\Gamma(t+s)=\frac{1}{t(k!)}\Gamma(t+k+1) \qquad t\neq 0,-1,-2,\ldots \tag{A2.3}$$

which can be established by induction, the first and fourth terms of (A2.2) are precisely equal to the first and third terms of (A2.1). Next, by interchanging the order of summations over s and l, the second term of (A2.2) may be written as

$$\sum_{l=1}^{k}\left(\sum_{s=0}^{k-l}\frac{(-1)^{k-l-s}}{s!(k-l-s)!}\right)E(\eta+l|m)y^{-2\eta-2l}. \tag{A2.4}$$

Since the sum over s appearing here is simply $\delta_{k,l}$, this term reduces to $E(\eta+k|m)$ $y^{-2\eta-2k}$; eventually, this will have to be compared with the second term of (A2.1), leading to the desired result (21).

To complete the proof, we have yet to establish the equality of the remaining terms of the two expressions. For this we note that the third term of (A2.2) simply drops out because the sum over s appearing there is equal to $\delta_{k,0}$ while our $E(\eta+k|m)$ are defined only for $k=1, 2, 3, \ldots$. Finally, we take a look at the 'remnant' terms involving summation over l. The summand in the case of (A2.2) is (see also equation (20))

$$(l^2+p)^{\eta+k-s-\frac{1}{2}m}-\sum_{r=0}^{k-s}\frac{1}{r!}\left(\frac{\partial^r}{\partial p^r}(l^2+p)^{\eta+k-s-\frac{1}{2}m}\right)_{p=0}p^r \tag{A2.5}$$

whose first part contributes

$$\tfrac{1}{2}\pi^{-m/2}\sum_{l(m)}{}'\left(\sum_{s=0}^{k}\frac{1}{s!}p^{-\eta-k+s}\Gamma(\tfrac{1}{2}m-\eta-k+s)(l^2+p)^{\eta+k-s-\frac{1}{2}m}\right) \tag{A2.6}$$

while the second part contributes

$$-\tfrac{1}{2}\pi^{-m/2}\sum_{l(m)}{}'\left[\sum_{s=0}^{k}\frac{1}{s!}p^{-\eta-k+s}\left(\sum_{r=0}^{k-s}\frac{(-1)^r}{r!}\Gamma(\tfrac{1}{2}m-\eta-k+s+r)l^{2(\eta+k-s-r)-m}p^r\right)\right]. \tag{A2.7}$$

Writing $r+s=t$, (A2.7) assumes the form

$$\begin{aligned}-\tfrac{1}{2}\pi^{-m/2}&\sum_{l(m)}{}'\left[\sum_{t=0}^{k}p^{-\eta-k+t}\Gamma(\tfrac{1}{2}m-\eta-k+t)l^{2(\eta+k-t)-m}\left(\sum_{r=0}^{t}\frac{(-1)^r}{(t-r)!r!}\right)\right]\\&=-\tfrac{1}{2}\pi^{-m/2}p^{-\eta-k}\Gamma(\tfrac{1}{2}m-\eta-k)\sum_{l(m)}{}'l^{2(\eta+k)-m}\end{aligned} \tag{A2.8}$$

because the sum over r here is simply $\delta_{t,0}$. Expressions (A2.6) and (A2.8) together reproduce the last term of (A2.1), which completes the proof of the desired relationship.

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